

Tunneling in two dimensional QCD

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Abstract

The spectral density for two dimensional continuum QCD has a non-analytic behavior for a critical area. Apparently this is not reflected in the Wilson loops. However, we show that the existence of a critical area is encoded in the winding Wilson loops: Although there is no non-analyticity or phase transition in these Wilson loops, the dynamics of these loops consists of two smoothly connected domains separated by the critical area, one domain with a confining behavior for large winding Wilson loops, and one (below the critical size) where the string tension disappears. We show that this can be interpreted in terms of a simple tunneling process between an ordered and a disordered state. In view of recent results by Narayanan and Neuberger this tunneling may also be relevant for four dimensional QCD.

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1 Introduction

Recently there appeared an interesting paper by Narayanan and Neuberger [1] where four dimensional Wilson operators for QCD on a lattice were shown to have a finite and nontrivial continuum limit. One of the surprising results is that the spectral density of eigenvalues of the Wilson loop develops a gap at infinite N for small size loops. This gap is closed when the loop size is increased sufficiently. The resulting eigenvalue distribution was compared to the corresponding density in two dimensional continuum QCD, and the agreement between four and two dimensions was very good.

The spectral density was computed by Durhuus and the author [2] from the Makeenko-Migdal equation. Later on the same result was obtained by Bassetto, Griguolo, and Vian [3] by direct summation of winding Wilson loops. One result is that for a critical size of the (dimensionless) area the spectral density becomes non-analytic. However, the winding Wilson loops are perfectly analytical as emphasized by Narayanan and Neuberger [1], so the existence of the critical loop size does not represent a phase transition in these loops. As a matter of fact, a superficial inspection of the Wilson loops defined on winding curves does not show any sign of a critical area. So the question is what is the meaning of the non-analyticity in the spectral density?

In this note we shall show that the critical area is indeed *encoded* in the Wilson loops with more than one winding. This encoding shows up as a smooth transition from different types of physics. At short distances the winding loops oscillate, and have a “non-stringy” behavior. For larger loops the string-like behavior is regained. We compare this behavior successfully to tunneling in ordinary non-relativistic quantum mechanics. It turns out that the critical size corresponds to the turning point, and the smaller sizes are analogous to the “classically allowed” region, whereas the confining larger loops correspond to the “classically forbidden” region. In view of the results obtained in ref. [1] a similar phenomenon most likely happens in four dimensions.

We emphasize that the idea of tunneling from an intermediate state with no confinement to a confining state is certainly not new. Neuberger [4] has discussed this several years ago, and it is also mentioned in ref. [1]. Our new point is that in contrast to what one would naively think by inspection of the winding Wilson loops, these have encoded information on a tunneling transition with a turning point exactly at the critical area.

In Section 2 we summarize the results on two-dimensional $N = \infty$ QCD. In Section 3 we develop asymptotic results for Wilson loops with a large number of windings, and in Section 4 we connect these loops with the gap in the spectral density. In Section 5 we develop the tunneling picture in the WKB approximation, and make some conclusions.

2 The gap and non-analyticity in the spectral density

We start by giving an overview of the results previously obtained [2, 3] in two dimensional QCD which are relevant for the following. The spectral density gives the distribution of the

continuous $N = \infty$ eigenvalues θ . From [2], eqs. (4.1) and (4.2), one easily finds

$$\theta = \arccos \left(\cosh y - A \frac{\sinh y}{2y} \right) + \frac{y}{\sinh y} \sqrt{1 - \cosh^2 y - A^2 \frac{\sinh^2 y}{4y^2} + A \frac{\sinh y \cosh y}{y}}, \quad (1)$$

where $y = -\pi A \rho_A(\theta)$, and where $A = g^2 N \times (\text{Area})$, which means that $A = 2k$, where k is the parameter used in [2]. The parameter A is a dimensionless area, defined such that the Wilson loop has the behavior

$$W = e^{-A/2}. \quad (2)$$

There exists a finite gap $\theta_c \leq \pi$ for $A < 4$, given by

$$\theta_c = \sqrt{A - \frac{A^2}{4}} + \arccos(1 - \frac{A}{2}). \quad (3)$$

For $A \geq 4$ the gap disappears, and the maximum value of θ is π .

Near the gap one has

$$\rho_A(\theta) \approx \frac{\sqrt{2(\theta_c - \theta)}}{\pi A (\frac{4}{A} - 1)^{1/4}}, \quad \theta \sim \theta_c. \quad (4)$$

Thus the derivative $\partial \rho_A(\theta)/\partial \theta$ becomes infinite at the gap.

Close to $A = 4$ this breaks down, and instead one gets

$$\rho_A(\theta) \approx \frac{1}{4\pi} \left(\frac{9\sqrt{3}}{2} (\pi - \theta) \right)^{1/3}. \quad (5)$$

For $A > 4$ the behavior near $\theta = \pi$ is smooth. The value y_c at $\theta = \pi$ is determined by

$$\cosh y_c - A \frac{\sinh y_c}{2y_c} = -1. \quad (6)$$

Expanding around $y = y_c$ with $y \approx y_c + \delta y$, we get

$$\delta y \approx (\pi - \theta)^2 \left[2 \left(1 - \frac{y_c}{\sinh y_c} \right)^2 \left(\left(1 + \frac{A}{2y_c^2} \right) \sinh y_c - \frac{A}{2y_c} \cosh y_c \right) \right]^{-1}. \quad (7)$$

From this we see that the derivative $\partial \rho/\partial \theta$ vanishes at $\theta = \pi$. For A large, $y_c \approx -A(1 - 2e^{-A/2})/2$, and one obtains

$$\rho_A(\theta) \approx \rho_A(\pi) + \frac{1}{2\pi} e^{-A/2} (\pi - \theta)^2 + \dots \quad (8)$$

For $A \rightarrow 4+$ we can easily find the small y_c from (6),

$$y_c \approx -\sqrt{3(A - 4)}. \quad (9)$$

Therefore the derivative $\partial \rho_A(\theta)/\partial A$ becomes infinite for $A \rightarrow 4$. In the following we shall show that this behavior reflects itself in the winding Wilson loops as tunneling with the turning point $A = 4$.

3 The Wilson loops with windings

The Wilson loop with n windings found by Kazakov and Kostov [7] is completely analytic in A ,

$$W^{(n)}(A) = \frac{1}{n} L_{n-1}^1(nA) e^{-nA/2}, \quad (10)$$

where L_{n-1}^1 is the Laguerre polynomial of type one,

$$L_{n-1}^1(nA) = \sum_{k=0}^{n-1} \frac{n!n^k}{(k+1)(k!)^2(n-k-1)!} (-A)^k. \quad (11)$$

The lack of analyticity of ρ for $A = 4$ obviously occurs due to the summation defining the spectral density,

$$\rho_A(\theta) = \frac{1}{2\pi} \left[1 + 2 \sum_{n=1}^{\infty} \cos(n\theta) W^{(n)}(A) \right]. \quad (12)$$

Also, from (10) it appears that $W^{(n)}$ has no reference whatsoever to the occurrence of the gap θ_c in the spectral density.

The non-analyticity and the existence of a gap for $A < 4$ is clearly related to the large n behavior of $W^{(n)}$, since if the sum in (12) is truncated, these phenomena do not occur. This is well known to several authors (for a recent reference, see [1]). However, we shall now show that $W^{(n)}(A)$ has a behavior which, at least for large n , reflects the existence of the gap θ_c . To this end we need the *uniform* large n asymptotic behavior of the Laguerre polynomial found by Erdélyi [5] from Laguerre's differential equation, and later also found from integral representations of the Laguerre polynomial [6]. There are two (partially overlapping) representations of these asymptotes, called the *Bessel form* and the *Airy form*. Applying Erdélyi's Bessel form to eq. (10) we get

$$W^{(n)}(A) \approx \frac{\sqrt{2}}{nA} \left(\frac{\sqrt{t-t^2} + \arcsin \sqrt{t}}{\sqrt{1/t-1}} \right)^{1/2} J_1 \left(2n(\sqrt{t-t^2} + \arcsin \sqrt{t}) \right) + \dots, \quad (13)$$

where $t = A/4$. We have checked numerically that the corrections “...” are extremely small even for moderate values of n , except for $A \approx 4$, where the approximation breaks down. However, the larger n is, the closer to $A = 4$ does the approximation work.

The Bessel form is thus valid for $A < 4$ and $A \neq 0$.

On the other hand, applying Erdélyi's Airy form, we obtain for $0 < t \leq 1$

$$W^{(n)}(A) \approx \frac{(-1)^{n-1} 2^{\frac{1}{2}} 3^{\frac{1}{6}}}{n^{\frac{4}{3}} A} \left(\frac{(\arccos \sqrt{t} - \sqrt{t-t^2})^{\frac{1}{3}}}{\sqrt{1/t-1}} \right)^{\frac{1}{2}} Ai \left(-(3n)^{\frac{2}{3}} (\arccos \sqrt{t} - \sqrt{t-t^2})^{\frac{2}{3}} \right). \quad (14)$$

For $t \geq 1$ we get from [5]

$$W^{(n)}(A) \approx \frac{(-1)^{n-1} 2^{\frac{1}{2}} 3^{\frac{1}{6}}}{n^{\frac{4}{3}} A} \frac{(\sqrt{t^2-t} - \ln(\sqrt{t} + \sqrt{t-1}))^{\frac{1}{6}}}{(1-1/t)^{\frac{1}{4}}} Ai \left((3n)^{\frac{2}{3}} (\sqrt{t^2-t} - \ln(\sqrt{t} + \sqrt{t-1}))^{\frac{2}{3}} \right). \quad (15)$$

The Airy function is defined by

$$Ai(z) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3}t^3 + zt\right)dt \quad (16)$$

and the Airy forms (14) and (15) can be expressed in terms of Bessel functions by means of the formulas

$$Ai(z) = \frac{1}{3}\sqrt{|z|} \left(J_{1/3}\left(\frac{2|z|^{3/2}}{3}\right) + J_{-1/3}\left(\frac{2|z|^{3/2}}{3}\right) \right) \quad \text{for } z < 0, \quad (17)$$

and

$$Ai(z) = \frac{1}{\pi}\sqrt{\frac{z}{3}} K_{1/3}\left(\frac{2z^{3/2}}{3}\right) \quad \text{for } z > 0. \quad (18)$$

It should be noticed that these Bessel functions are different from the J_1 Bessel function in eq. (13).

4 Connection between the gap parameter θ_c and $W^{(n)}$

In this Section we shall show that in spite of their appearance the $W^{(n)}$'s with $n > 1$ know about the gap, and have quite different behaviors on the two sides of $A = 4$.

If we consider $A < 4$ we see that the argument of the Bessel function (13) is similar to the gap parameter θ_c given in eq. (3). The connection is easily found,

$$\begin{aligned} \sqrt{t - t^2} + \arcsin \sqrt{t} &= \frac{1}{2} \left(\sqrt{A - \frac{A^2}{4}} + 2 \arccos \sqrt{1 - \frac{A}{4}} \right) \\ &= \frac{1}{2} \left(\sqrt{A - \frac{A^2}{4}} + \arccos(1 - \frac{A}{2}) \right) = \frac{1}{2} \theta_c. \end{aligned} \quad (19)$$

Therefore eq. (13) can be written

$$W^{(n)}(A) = \frac{\theta_c^{1/2}}{nA(4/A - 1)^{1/4}} J_1(n\theta_c) + \dots \quad (20)$$

Similarly, we also have

$$\arccos \sqrt{t} - \sqrt{t - t^2} = \frac{1}{2}(\pi - \theta_c), \quad (21)$$

so we can rewrite eq. (14),

$$W^{(n)}(A) \approx \frac{(-1)^{n-1} 2^{1/3} 3^{1/6} (\pi - \theta_c)^{1/6}}{n^{4/3} A (4/A - 1)^{1/4}} Ai\left(-\left(\frac{3n}{2}\right)^{2/3} (\pi - \theta_c)^{2/3}\right). \quad (22)$$

By means of eq. (17) this can also be written in terms of Bessel functions,

$$W^{(n)}(A) \approx \frac{(-1)^{n-1} (\pi - \theta_c)^{1/2}}{\sqrt{3} nA (4/A - 1)^{1/4}} \left[J_{1/3}(n(\pi - \theta_c)) + J_{-1/3}(n(\pi - \theta_c)) \right]. \quad (23)$$

The various Bessel functions in these expressions oscillate, and the oscillations are governed entirely by the gap parameter, either θ_c directly as in (20), or else by the length of the gap $\pi - \theta_c$ as in (23). Thus, the Wilson loops with windings $n > 1$ actually know about the gap parameter.

We have checked numerically that (20) and (23) give a very precise representation of the exact expression for $W^{(n)}$ in eq. (10). This holds for surprisingly low values of n , especially as far as the Airy form (23) is concerned. With the Bessel form (20) there are deviations close to $A = 4$, but one can get closer by taking large n 's, e.g. $n = 10$.

When n is sufficiently large, and θ_c or $\pi - \theta_c$ are not too small, we can use the well known asymptotic representations for the Bessel functions to rewrite (20) and (23),

$$W^{(n)}(A) \approx \frac{\sqrt{2}}{\sqrt{\pi} n^{3/2} A (4/A - 1)^{1/4}} \cos(n\theta_c - \frac{3\pi}{4}) + \dots \quad (24)$$

Although the Bessel functions are different in eqs. (20) and (23), after use of the addition theorem for cosines they lead to the same expression (24), as one would expect. The asymptotic expression (24) can be derived in a straightforward manner by the saddle point method from the integral representation of the Laguerre polynomial,

$$L_{n-1}^1(nA) = \int_C \frac{dt}{2\pi i} e^{-nAt} \left(1 + \frac{1}{t}\right)^n, \quad (25)$$

where the contour C encloses the origin.

The asymptotic expression (24) gives good results if A is not too close to 4. As an example, let us check the zeros predicted by (24) against the zeros in $L_{n-1}^1(nA)$ for the rather moderate value $n = 5$. The $\theta_c(A)$'s which correspond to the zeros in $L_4^1(5A)$ are

$$0.766, \quad 1.403, \quad 2.034, \quad \text{and} \quad 2.662 \quad (26)$$

The zeros predicted by (24) are

$$0.785, \quad 1.414, \quad 2.042, \quad \text{and} \quad 2.670. \quad (27)$$

The largest discrepancy is of order three per cent. With $n > 5$ the accuracy improves, of course.

We now turn to $A > 4$, where we have eq. (15). By use of (18) we can rewrite this expression as

$$\begin{aligned} W^{(n)}(A) \approx & \frac{(-1)^{n-1} \sqrt{2}}{\pi n A (1 - 4/A)^{1/4}} \left(\frac{A}{4} \sqrt{1 - \frac{4}{A}} - \ln\left(\frac{\sqrt{A}}{2} + \sqrt{\frac{A}{4} - 1}\right) \right)^{1/2} \\ & \times K_{1/3}\left(\frac{nA}{2} \left(\sqrt{1 - \frac{4}{A}} - \frac{4}{A} \ln\left(\frac{\sqrt{A}}{2} + \sqrt{\frac{A}{4} - 1}\right)\right)\right). \end{aligned} \quad (28)$$

By means of the asymptotic expansion of the K -function this gives ($A > 4$)

$$W^{(n)}(A) \approx \frac{(-1)^{n-1}}{\sqrt{2\pi} n^{3/2} A (1 - 4/A)^{1/4}} \left(\frac{\sqrt{A}}{2} + \sqrt{\frac{A}{4} - 1} \right)^{2n} \exp \left[-\frac{nA}{2} \sqrt{1 - \frac{4}{A}} \right]. \quad (29)$$

This result can also be derived by the saddle point method from the integral representation (25). For $A \rightarrow \infty$ eq. (29) leads to

$$W^{(n)}(A) \approx \frac{(-A)^{n-1} e^n}{\sqrt{2\pi} n^{3/2}} e^{-nA/2}. \quad (30)$$

This is precisely the expression one would get from the exact formula (10) by taking into account only the first term in the Laguerre polynomial (11) and using Stirling's formula to estimate the factorials for $n \rightarrow \infty$.

It is easy to check that eqs.(23) and (29) give good numerical results for $A \geq 4$. The same is not true for (30), which is only valid for A 's somewhat larger than 4.

The turning point region around $A = 4$ can be approximated by simpler expressions. The results depend on how close A is to 4. If $A - 4 \ll n^{-2/3}$ one finds [5]

$$W^{(n)}(A) \approx \frac{(-1)^{n-1}}{(2n)^{4/3} 3^{2/3} \Gamma(2/3)}, \quad (31)$$

whereas for $A - 4 \ll n^{-2/5}$ the approximation is

$$W^{(n)}(A) \approx \frac{(-1)^{n-1}}{(2n)^{4/3}} Ai(n^{2/3}(A - 4)/(22^{1/3})). \quad (32)$$

These two expressions agree if $A = 4$ because $Ai(0) = 1/(3^{2/3}\Gamma(2/3))$.

5 On the physical meaning of the gap and non-analyticity: Tunneling

The results obtained in the previous section show that the Wilson loop $W^{(n)}$ with two or more windings is influenced by the gap and the $A = 4$ behavior. There are two different physical regions, but they are smoothly connected: In the region $A < 4$ there is no area behavior, but oscillations. In the second case, there is an area behavior. This is in contrast to $W^{(1)}$, where there is always an area behavior $e^{-A/2}$.

One may wonder why the area behavior is not present for A less than 4, because the Wilson loop (10) contains the factor $e^{-nA/2}$, which has a large negative exponent for large n , even if A is moderate. The answer to this point is that the Laguerre polynomial has large coefficients for large n . For example, the leading term is of order $A^n e^n$, which for moderate A can win over $e^{-nA/2}$.

The non-analytic behavior of the spectral density must be a non-smooth reflection of the smooth transition between the two physically different regions, separated by $A = 4$. The Wilson loops $W^{(n)}$ are analytic in A , as follows trivially from (10). However, if we think in terms of a string description of the dynamics, we see that the Wilson loop with $n > 1$ is able to go from the "evident" string at larger A to something which does not look "stringy" at smaller A 's.

Let us look at the eigenvalue gap parameter $\theta_c(A)$ as a coordinate, which is compact but continuous (thanks to $N = \infty$) in the region $A < 4$. For $A > 4$ we can instead take the

argument of the Bessel K -function as a coordinate. Let us denote this set of “coordinates” by ξ ,

$$\xi_1 = \pi - \theta_c(A) \quad (A < 4), \quad \xi_2 = \sqrt{A^2/4 - A} - 2 \ln(\sqrt{A}/2 + \sqrt{A/4 - 1}) \quad (A > 4). \quad (33)$$

Then the (smooth) transition that occurs at $A = 4$ looks very much like tunneling, where $A < 4$ is the “classically allowed region”, whereas $A > 4$ is the region which is “classically forbidden”, and $A = 4$ is the turning point. The Bessel functions we found are indeed similar to those wave functions that are encountered in the WKB approximation.

To see the connection to the WKB approximation in details, we notice that ξ should be represented by an integral over “momenta” k_L and k_R . The integrals are

$$\int_A^4 k_L dx = n \xi_1 \quad \text{and} \quad \int_4^A k_R dx = n \xi_2, \quad (34)$$

where we consider n to be a fixed number. From eqs. (33) we get

$$k_L = \frac{n}{2} \sqrt{\frac{4-A}{A}}, \quad k_R = \frac{n}{2} \sqrt{\frac{A-4}{A}}. \quad (35)$$

Near the turning point $A = 4$ we therefore see that the squares of the momenta are linear, $\propto 4 - A$ or $A - 4$. Therefore the WKB-solution on the left should be expressed in terms of $\xi_1^{1/2} J_{\pm 1/3}(n\xi_1)$ and to the right by $\xi_2^{1/2} I_{\pm 1/3}(n\xi_2)$ (see e.g. ref. [8], pp. 188-190). Taking into account that

$$K_{1/3}(x) = \frac{\pi}{\sqrt{3}} [I_{-1/3}(x) - I_{1/3}(x)]. \quad (36)$$

this is precisely what we obtained using Erdélyi’s approximations. This even applies to the prefactors

$$(4/A - 1)^{-1/4} \quad \text{or} \quad (1 - 4/A)^{-1/4}, \quad (37)$$

which correspond to the prefactors $k_L^{-1/2}$ or $k_R^{-1/2}$ occurring in the WKB expressions [8].

It should be noticed that even without using the large n approximate results, the Wilson loops (10) for $n > 1$ clearly distinguish between values of A larger or smaller than 4. This for example follows because the Laguerre polynomials $L_{n-1}^a(x)$ have all their zeros in the region $0 < x < \nu$, where $\nu = 4n + 2a - 2$, and outside this interval the polynomials are monotonic [5]. In our case (take $x = nA$) this condition for oscillations means $0 < A < 4$, so the zeros are all below $A = 4$, and in this region the Wilson loops (10) oscillate. There is therefore always a non-stringy behavior below $A = 4$ for $n \geq 2$. In this connection it is amusing to notice that if we define a “wave function” ψ

$$\psi_n(A) = A W^{(n)}(A), \quad n > 1, \quad (38)$$

then from (10) and the differential equation for the Laguerre polynomials it follows that ψ satisfies the linear “Schrödinger equation”

$$\frac{d^2 \psi_n(A)}{dA^2} + \frac{n^2}{4} \left(\frac{4}{A} - 1 \right) \psi_n(A) = 0. \quad (39)$$

In this equation no approximation has been used. Eq. (39) clearly shows the existence of the turning point at $A = 4$. The approximate results discussed in the preceding sections were actually derived from (39), see ref. [5].

The physics behind the domains separated by $A = 4$ is indicated by the spectral density. In the extreme case where A is small, the eigenvalues θ are centered around $\theta = 0$, and in the other extreme where A is large, the eigenvalues are almost uniformly distributed. Thus, the domain to the right of $A = 4$ corresponds approximately to a completely disordered state, whereas the domain to the left of $A = 4$ is a much more ordered state, at least as far as the eigenvalues are concerned. Therefore the confinement is related to randomness, and the approximately ordered state leads to an “evaporation” of the effective string tension. This evaporation happens to the left and even somewhat to the right of $A = 4$, see the exponent in eq. (29).

The coordinates $\xi_{1,2}$ represent a new degree of freedom, which perhaps may be interpreted as an extra dimension, following ideas by Das and Jevicki [9] and by Brower and collaborators [10]. What this adds up to is that $\xi_{1,2}$ might play a dynamical role in an effective string theory in three dimensions, where the action provides a barrier, which produces tunneling in the new dimension. One difficulty with this picture is that this tunneling does not occur for $W^{(1)}$.

6 Conclusions

The main conclusion of this note is that the behavior of $W^{(n)}(A)$ as a function of A physically represents a smooth tunneling between a short distance ($A < 4$) and a large distance state ($A > 4$). In view of the results obtained by Narayanan and Neuberger [1] on the relation between four and two dimensional QCD this may be true not only in two, but also in four dimensions. The $A < 4$ region may be considered as a transient state where there is no “obvious” string-picture. The tunneling to the disordered confined state has undoubtedly different causes in different dimensions (instantons, monopoles, vortex lines,...). In four dimensions the asymptotic freedom short-distance region would be hooked on to the approximately ordered state, which is then intermediate between freedom and confinement. Of course, such a picture does not exclude that there exists a sufficiently intelligent string theory which can accomodate this behavior, as advocated by several authors, most recently by Brower [10]. However, it is of some interest that the $W^{(n)}$ ’s have at least some knowledge of the complexity involved at different scales.

The connection between disorder and large distance dimensional reduction was actually argued a long time ago [11] to be connected to confinement, because e.g. four dimensional QCD could reduce to two dimensional QCD at large distances as a consequence of the disorder. The results obtained in [1] precisely means the existence of a disordered vacuum at large distances in four dimensions. The phenomenon disorder \rightarrow reduced dimension is known to occur in simple condensed matter models [12], e.g. a scalar field coupled to a random Gaussian field. In this relatively simple case the disorder reduces the effective dimension D by two, $D \rightarrow D - 2$. The complexity of QCD means that the results and the methods from solid state physics have not been generalized to QCD, and it is not known

by how much the dimension effectively reduces. Thus it may be that both four and three dimensions effectively reduce to two dimensions. In QCD the disordered field should be generated dynamically in the “true” vacuum, and not put in externally, as in the condensed matter case. The complexity of QCD is a good reason to study this problem by using lattice QCD. It would be interesting to investigate on a lattice whether the tunneling transition found in the present paper also occurs for (winding) Wilson loops in four dimensions. The similarity of the spectral density in four and two dimensions found in [1] indicates that this would be the case.

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